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Parametric Approximation

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1. INTRODUCTION

Let V be a nonempty subset of the normed linear space X. For any $x \in X$, the (possibly empty) *set of best approximations*, or nearest points, to x from V is defined by

$$P_V(x) = \{v \in V \mid \| x - v \| = d(x, V)\},$$

where $d(x, V) = \inf\{\| x - v \| \mid v \in V\}$. The (set-valued) mapping $P_V : X \to 2^V$ is called the *metric projection* onto V.

Clearly, the set $P_V(x)$ depends on the point x, the set V, the norm $\| \cdot \|$, and the linear space X. It is natural to ask if $P_V(x)$ varies "continuously" relative to x, V, $\| \cdot \|$, and X.

The question of the dependence of $P_V(x)$ on x is essentially the problem of the continuity of the metric projection P_V. This problem has been considered by many authors. Partial surveys may be found, for example, in Vlasov [17] and Singer [15].

The question as to how $P_V(x)$ depends on the norm $\| \cdot \|$ has been considered rarely (see, e.g., Kripke [6] and Rice [13; the Pólya algorithm, p. 8]).

The question of the dependence of $P_V(x)$ on the linear space X has only been considered in some special cases (see, e.g., Kirchberger [5], Maehly and Witzgall [7], Nitsche [9], and Chui *et al.* [2]).

However, the problem of the dependence of $P_V(x)$ on V (for a fixed x)

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does not seem to have been considered up to now. This question seems important since, for example, when approximating with spline functions, it is of interest to know how the set of best approximations to a given function depends on the knots which define the splines. In practice, the knots cannot be specified exactly, but only up to some error. It is reasonable to ask if the best approximations change "continuously" as the error tends to zero.

Another reason for wanting to study the behavior of $P_V(x)$ as V varies is that many important *nonlinear* approximating subsets V (e.g., the rational functions, exponential sums, or the spline functions with free knots) can be expressed as the union of *linear subspaces*:

$$V = \bigcup \{V_a \mid a \in A\}$$

for some $A \subset \mathbb{R}^n$, where each V_a is a (linear) subspace. For example, the rational functions in $C[0, 1]$ can be written as

$$R_n{}^m[0, 1] = \bigcup \{V_a \mid a \in A\},$$

where

$$V_a = \left\{\left(\sum_0^m c_i t^i\right)\left(\sum_0^n a_j t^j\right)^{-1} \middle| (c_0, c_1, ..., c_m) \in \mathbb{R}^{m+1}\right\}$$

and

$$A \equiv \left\{a = (a_0, a_1, ..., a_n) \in \mathbb{R}^{n+1} \middle| \sum_0^n a_i t^i > 0 \text{ for all } t \in [0, 1]\right\}.$$

This suggests the following procedure for finding best approximations to a given function x from V. Fix a parameter $a \in A$ and determine a best approximation to x from the subspace V_a. Then choose a new parameter $\tilde{a} \in A$ so that the above *linear* approximation problem yields a "better" approximation to x from $V_{\tilde{a}}$. The question is now whether such an algorithm exists which gives a sequence of elements converging to a best approximation to x from V. A natural first step is to determine if best approximations to x from V_a vary "continuously" relative to the parameter a.

In this paper we consider the following general situation. Let A be a topological space (set of "parameters"), X a normed linear space, $x \in X$, and for each $a \in A$, let V_a be a nonempty subset of X. The *parameter mapping* is the (generally set-valued) mapping $a \mapsto P_{V_a}(x)$ from A into the collection of bounded subsets of X. We will be interested in how $P_{V_a}(x)$ varies with a.

In Section 2, we include some results of a general nature concerning semicontinuity properties of the parameter mapping in the case that each V_a is a finite-dimensional flat. The main result here is Theorem 2.5 which gives a sufficient condition under which the parameter mapping is upper semicontinuous. On the other hand we give an example which shows that even in a two-dimensional polyhedral space, where the range $\{V_a \mid a \in A\}$ consists of all one-dimensional subspaces, the parameter mapping is neither lower semicontinuous nor admits a continuous selection (Example 2.7). This is in striking contrast to the behavior of the metric projection itself which, in this situation, is always lower semicontinuous and admits a continuous (even linear!) selection. In this connection, we should mention that the metric projection can be regarded, in a certain sense, as a special parameter mapping because of the formula $P_V(x) = x + P_{V-\{x\}}(0)$ for each $x \in X$. From this it follows that any semicontinuity properties of the parameter mapping $x \mapsto P_{V-x}(0)$ are valid for the metric projection $x \mapsto P_V(x)$, and vice versa.

In Section 3, we consider parametric approximation by weak Chebyshev subspaces of $C[a, b]$. In contrast to Example 2.7, we show that the parameter mapping for a certain class of weak Chebyshev subspaces does admit a continuous selection (Theorem 3.3). Furthermore, we prove that the natural parameter mapping for spline subspaces is always upper semicontinuous (Proposition 3.6), but in general not lower semicontinuous (Proposition 3.8); however, it *does* admit a continuous selection for a special class of spline subspaces (Theorem 3.9).

2. Some General Results

If (X, d) is a metric space, the *Hausdorff metric* h on the collection of all nonempty closed and bounded subsets $\mathscr{B}(X)$ of X is defined by

$$h(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

where $d(x, Y) = \inf\{d(x, y) \mid y \in Y\}$.

2.1 Lemma. *Let A and B be nonempty closed and bounded subsets of a metric space (X, d) and let $x \in X$. Then*

$$|d(x, A) - d(x, B)| \leqslant h(A, B).$$

In particular, if $\mathscr{B}(X)$ is topologized by the Hausdorff metric, the map $d(x, \cdot)$: $\mathscr{B}(X) \to \mathbb{R}$ is Lipschitz continuous.

Proof. Given $\epsilon > 0$, choose $a \in A$ and $b \in B$ so that $d(x, a) < d(x, A) + \epsilon/2$ and $d(a, b) < d(a, B) + \epsilon/2$. Then

$$\begin{aligned} d(x, B) &\leqslant d(x, b) \leqslant d(x, a) + d(a, b) \\ &< d(x, A) + d(a, B) + \epsilon \\ &\leqslant d(x, A) + h(A, B) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary,

$$d(x, B) - d(x, A) \leqslant h(A, B).$$

Interchanging the roles of A and B, we obtain the result. ▮

2.2 DEFINITION (See, e.g., Hahn [3]). Let A be a topological space, X a metric space, and $\mathscr{B}(X)$ the collection of all nonempty closed and bounded subsets of X. Let $a_0 \in A$. A (set-valued) mapping $F: A \to \mathscr{B}(X)$ is called:

(1) *lower semicontinuous* (l.s.c) *at* a_0 if for each open set $W \subset X$ with $F(a_0) \cap W \neq \varnothing$, there is a neighborhood U of a_0 (in A) such that $F(a) \cap W \neq \varnothing$ for all $a \in U$;

(2) *upper semicontinuous* (u.s.c) *at* a_0 if each open set $W \subset X$ with $F(a_0) \subset W$, there is a neighborhood U of a_0 (in A) such that $F(a_0) \subset W$ for each $a \in U$;

(3) *Hausdorff continuous at* a_0 if for each $\epsilon > 0$, there is a neighborhood U of a_0 (in A) such that $h(F(a_0), F(a)) < \epsilon$ for each $a \in A$.

F is called lower semicontinuous (l.s.c), upper semicontinuous (u.s.c), or Hausdorff continuous, if F is (respectively) l.s.c., u.s.c., or Hausdorff continuous at each point of X.

Note that if F is "singleton-valued" (i.e., $F(a)$ is a point for each $a \in A$), each of these three conditions reduces to the usual definition of continuity of the mapping $a \mapsto F(a)$.

A *continuous selection* for F is a continuous mapping $s: A \to X$ such that $s(a) \in F(a)$ for each $a \in A$.

2.3 LEMMA. *Let A be a topological space, X a metric space $x \in X$, and for each $a \in A$, let K_a be a nonempty compact subset of X. If the mapping $a \mapsto K_a$ (from A into $\mathscr{B}(X)$) is Hausdorff continuous at a_0, then the parameter mapping $a \mapsto P_{K_a}(x)$ (from A into $\mathscr{B}(X)$) is upper semicontinuous at a_0.*

Proof. If not, there is an open set $W \subset X$ with $P_{K_{a_0}}(x) \subset W$, and a net (a_δ) in A with $a_\delta \to a_0$ such that $P_{K_{a_\delta}}(x) \backslash W \neq \varnothing$ for each δ. Choose

$$y_\delta \in P_{K_{a_\delta}}(x) \backslash W \qquad \text{for each } \delta. \tag{*}$$

Since the map $a \mapsto K_a$ is Hausdorff continuous, $h(K_{a_\delta}, K_{a_0}) \to 0$. But this implies that $\sup_{k \in K_{a_\delta}} d(k, K_{a_0}) \to 0$. Hence $d(y_\delta, K_{a_0}) \to 0$. By compactness, we can choose $k_\delta{}^0 \in K_{a_0}$ so that $d(y_\delta, k_\delta{}^0) = d(y_\delta, K_{a_0})$. Again by compactness, there exist a subnet $(k_\gamma{}^0)$ of $(k_\delta{}^0)$ and $k^0 \in K_{a_0}$ such that $d(k_\gamma{}^0, k^0) \to 0$. Thus

$$d(y, k^0) \leqslant d(y_\gamma, k_\gamma{}^0) + d(k_\gamma{}^0, k^0) \to 0.$$

Using Lemma 2.1, we deduce that

$$d(x, k^0) = \lim d(x, y_\gamma) = \lim d(x, K_{a_\gamma}) = d(x, K_{a_0})$$

so $k^0 \in P_{K_{a_0}}(x) \subset W$. Hence $y_\gamma \in W$ eventually, which contradicts (*). ∎

A *flat* in a linear space X is any set of the form $V = M + x$, where M is a linear subspace and $x \in X$. That is, a flat is a translation of a linear subspace. An *n-dimensional flat* is the translation of an n-dimensional subspace.

As an application of Lemma 2.3 to normed linear spaces, we prove

2.4 PROPOSITION. *Let X be a normed linear space, $x \in X$, A a topological space, and for each $a \in A$, let V_a be a finite-dimensional flat in X and let c_a be any constant such that $d(x, V_a) + \| x \| \leqslant c_a$. (E.g., if V_a is actually a subspace, then any constant $c_a \geqslant 2 \| x \|$ works.) Let*

$$\tilde{V}_a = \{v \in V_a \mid \| v \| \leqslant c_a\}.$$

If the mapping $a \mapsto \tilde{V}_a$ is Hausdorff continuous on A, then the parameter mapping $a \mapsto P_{V_a}(x)$ is upper semicontinuous on A.

Proof. Since $\tilde{V}_a$ is a closed and bounded subset of a finite-dimensional flat, $\tilde{V}_a$ is compact and Lemma 2.3 implies that the map $a \mapsto P_{\tilde{V}_a}(x)$ is u.s.c. But if $v \in P_{V_a}(x)$, then

$$\| v \| \leqslant \| v - x \| + \| x \| = d(x, V_a) + \| x \| \leqslant c_a .$$

That is, $v \in \tilde{V}_a$. Hence $P_{\tilde{V}_a}(x) = P_{V_a}(x)$ and the result follows. ∎

It may not always be easy to ascertain whether or not the mapping $a \mapsto \tilde{V}_a$ of Proposition 2.4 is Hausdorff continuous; however, the following theorem provides a useful alternate condition that is often easy to verify in the applications.

2.5 THEOREM. *Let X be a normed linear space, A a topological space, N a fixed positive integer, and for each $a \in A$, let V_a be an N-dimensional subspace of X. If V_a has a basis $\{x_1{}^a, x_2{}^a, \ldots, x_N{}^a\}$ which is continuous in a (i.e., $a_\delta \to a$ implies $\| x_i^{a_\delta} - x_i{}^a \| \to 0$ for $i = 1, 2, \ldots, N$), then for each $x \in X$ the mapping $a \mapsto \tilde{V}_a$ is Hausdorff continuous on A (where $\tilde{V}_a$ is defined as in Proposition 2.4 with $c_a = 2 \| x \|$). In particular, the parameter mapping $a \mapsto P_{V_a}(x)$ is upper semicontinuous on A.*

Proof. The key to our proof is the

LEMMA. *Under the hypothesis of the theorem, let $a_\delta \to a_0$ and $v_\delta \in \tilde{V}_{a_\delta} \equiv$*

$\{v \in V_{a_\delta} \mid \| v \| \leqslant 2 \| x \|\}$. *Then there exists a subnet of* (v_δ) *which converges to an element of* $\tilde{V}_{a_0}$.

Proof of lemma. Let $v_\delta = \sum_{i=1}^{N} \alpha_i^\delta x_i^{a_\delta}$. We first show that for $i = 1, 2,..., N$, the net (α_i^δ) is eventually bounded. If not, then by passing to a subnet and reindexing if necessary, we may assume

$$| \alpha_1^\delta | \geqslant | \alpha_i^\delta | \qquad (i = 1, 2,..., N)$$

and $0 < | \alpha_1^\delta | \to \infty$. Since $| \alpha_i^\delta / \alpha_1^\delta | \leqslant 1$, by passing to a further subnet, we may assume that $\alpha_i^\delta / \alpha_1^\delta \to \beta_i \in \mathbb{R}$ $(i = 1, 2,..., N)$. Then

$$\left\| x_1^{a_0} + \sum_2^N \beta_i x_i^{a_0} \right\| = \lim_\delta \left\| x_1^{a_\delta} + \sum_2^N \frac{\alpha_i^\delta}{\alpha_1^\delta} x_i^{a_\delta} \right\|$$

$$= \lim_\delta \frac{1}{| \alpha_1^\delta |} \| v_\delta \| \leqslant \limsup_\delta \frac{2 \| x \|}{| \alpha_1^\delta |} = 0.$$

Thus $x_1^{a_0} + \sum_2^N \beta_i x_i^{a_0} = 0$, which contradicts the linear inedependence of $\{x_1^{a_0}, x_2^{a_0},..., x_N^{a_0}\}$. Thus (α_i^δ) is eventually bounded for each $i \in \{1, 2,..., N\}$. Hence there is a subnet so that

$$\alpha_i^\gamma \to \alpha_i \in \mathbb{R} \qquad (i = 1, 2,..., N).$$

Thus

$$v_\gamma = \sum_1^N \alpha_i^\gamma x_\gamma^{a_\gamma} \to \sum_1^N \alpha_i x_i^{a_0} \equiv v_0 \in V_{a_0}.$$

Since $\| v_\gamma \| \leqslant 2 \| x \|$ for all γ, $\| v_0 \| \leqslant 2 \| x \|$. That is, $v_0 \in \tilde{V}_{a_0}$. This proves the lemma.

To prove the theorem, let $a_\delta \to a_0$. We first show that $h(\tilde{V}_{a_0}, \tilde{V}_{a_\delta}) \to 0$. That is, we must show

$$\text{(a)} \quad \sup_{v \in V_{a_\delta}} d(v, \tilde{V}_{a_0}) \to 0$$

and

$$\text{(b)} \quad \sup_{v \in V_{a_0}} d(v, \tilde{V}_{a_\delta}) \to 0.$$

If (a) were false, one could choose a subnet (a_β), $v_\beta \in \tilde{V}_{a_\beta}$, and $\epsilon > 0$ so that $d(v_\beta, \tilde{V}_{a_0}) \geqslant \epsilon$ for each β. By the lemma, there is a subnet (v_γ) of (v_β) and $v_0 \in \tilde{V}_{a_0}$ so that $v_\gamma \to v_0$. Thus

$$0 = d(v_0, \tilde{V}_{a_0}) = \lim_\gamma d(v_\gamma, \tilde{V}_{a_0}) \geqslant \epsilon$$

which is absurd.

If (b) were false, one could choose a subnet (a_β) of (a_δ), $v_\beta{}^0 \in \tilde{V}_{a_0}$, and $\epsilon > 0$ so that

$$d(v_\beta{}^0, V_{a_\beta}) \geqslant \epsilon \qquad \text{for each } \beta. \tag{**}$$

By compactness of $\tilde{V}_{a_0}$, and passing to a subnet if necessary, we may assume that

$$v_\beta{}^0 \equiv \sum_1^N \alpha_i{}^\beta x_i^{a_0} \to v^0 \equiv \sum_1^N \alpha_i x_i^{a_0}.$$

By the equivalence of norms in the finite-dimensional space V_{a_0}, it follows that $\alpha_i{}^\beta \to \alpha_i$ $(i = 1, 2,..., N)$. Define $v_\beta = \sum_1^N \alpha_i{}^\beta x_i^{a_\beta}$ and set

$$\begin{aligned} \tilde{v}_\beta &= v_\beta && \text{if} \quad \| v_\beta \| \leqslant 2 \| x \|, \\ &= \frac{\| v^0 \|}{\| v_\beta \|} v_\beta && \text{if} \quad \| v_\beta \| > 2 \| x \|. \end{aligned}$$

Since $v_\beta \in V_{a_\beta}$, it follows that $\tilde{v}_\beta \in \tilde{V}_{a_\beta}$. Also, since

$$\begin{aligned} \| v_\beta - v^0 \| &\leqslant \| v_\beta - v_\beta{}^0 \| + \| v_\beta{}^0 - v^0 \| \\ &\leqslant \sum_1^N | \alpha_i{}^\beta | \, \| x_i^{a_\delta} - x_i^{a_0} \| + \| v_\beta{}^0 - v^0 \| \to 0, \end{aligned}$$

it follows that $\| \tilde{v}_\beta - v^0 \| \to 0$. Hence

$$d(v_\beta{}^0, \tilde{V}_{a_\beta}) \leqslant \| v_\beta{}^0 - \tilde{v}_\beta \| \leqslant \| v_\beta{}^0 - v^0 \| + \| v^0 - \tilde{v}_\beta \| \to 0$$

which contradicts (**). This proves that $h(\tilde{V}_{a_0}, \tilde{V}_{a_\delta}) \to 0$. That is, $a \mapsto \tilde{V}$ is Hausdorff continuous on A. The last statement now follows from Proposition 2.4. ▮

We give now a few representative applications of Theorem 2.5.

2.6. Some Application

(1) *Best Approximation Using a Weight Function*

Let $X = C[0, 1]$, $x \in X$, $\{v_1, v_2,..., v_n\}$ a linearly independent subset of X, $V = \text{span}\{v_1, v_2,..., v_n\}$, and let A be a topological space. Let w: $A \to C[0, 1]$ be a continuous function such that $(w(a))(t) \neq 0$ for each $t \in [0, 1]$. It is easy to see that the vectors

$$x_i{}^a = w(a) \cdot v_i \qquad (i = 1, 2,..., n)$$

are linearly independent in X (for each $a \in A$) so $V_a = \text{span}\{x_1^a, x_2^a, \dots, x_n^a\}$ is n dimensional for each $a \in A$. Further, the functions $a \mapsto x_i^a$ $(i = 1, 2, \dots, n)$ are continuous on A. Hence by Theorem 2.5 the parameter mapping $a \mapsto P_{V_a}(x)$ is upper semicontinuous.

(2) *Best Approximation by Generalized Rational Functions*

Let T be a compact Hausdorff space, $X = C(T)$, and let $\{u_1, u_2, \dots, u_m\}$ and $\{v_1, v_2, \dots, v_n\}$ be two linearly independent subsets of X. Let

$$A = \left\{ a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n \;\Big|\; \sum_1^n a_i v_i(t) > 0 \text{ for all } t \in T \right\}$$

and, for each $a = (a_1, a_2, \dots, a_n) \in A$, define

$$V_a = \left\{ \left(\sum_1^n a_i v_i \right)^{-1} \left(\sum_1^m b_i u_i \right) \;\Big|\; (b_1, b_2, \dots, b_m) \in \mathbb{R}^m \right\}.$$

Then V_a is an m-dimensional subspace of $C(T)$ having the basis vectors

$$\frac{u_i}{\sum_{j=1}^n a_j v_j} \qquad (i = 1, 2, \dots, m).$$

Further, the mappings

$$a \mapsto \frac{u_i}{\sum_1^n a_j v_j} \qquad (i = 1, 2, \dots, n)$$

are continuous on A. Thus, by Theorem 2.5, the parameter mapping

$$a \mapsto P_{V_a}(x)$$

is upper semicontinuous on A (for each $x \in C(T)$).

(3) *Best Approximation by Exponential Sums*

Let $X = C[0, 1]$ and for each

$$a = (a_1, a_2, \dots, a_n) \in A = \{(a_1, a_2, \dots, a_n) \in \mathbb{R}^n \mid a_1 < a_2 < \cdots < a_n\}$$

define

$$V_a = \left\{ \sum_1^n \alpha_i e^{a_i t} \;\Big|\; (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n \right\}.$$

Then V_a is an n-dimensional subspace of $C[0, 1]$ and the mappings

$$a \mapsto e^{a_i t} \qquad (i = 1, 2, \dots, n)$$

are continuous on A. By Theorem 2.5, for each $x \in C[0, 1]$, the parameter mapping $a \mapsto P_{V_a}(x)$ is upper semicontinuous on A.

It is known that in normed linear space with the Property (P) of Brown [1], the metric projection onto any finite-dimensional subspace is lower semicontinuous, and hence (by the Michael selection theorem [8]) has a continuous selection. Therefore it is natural to conjecture that, with the hypothesis of Theorem 2.5 and the additional assumption that X has property (P), the parameter map $a \mapsto P_{V_a}(x)$ is lower semicontinuous. The following simple counterexample is then perhaps surprising. In fact, the parameter map of this example does not even have a continuous selection.

2.7 COUNTEREXAMPLE. Let X denote the plane with the maximum norm: $x = (x(1), x(2)) \in \mathbb{R}^2$, $\| x \| = \max\{| x(1), | x(2)|\}$. Then X is a polyhedral space so it has property (P) [1]. Let A denote the interval $[-1, 1]$ (with the usual topology), and for each $a \in A$, let $x_1{}^a = (1, a) \in X$ and

$$V_a = \operatorname{span}\{x_1{}^a\} \equiv \{\alpha x_1{}^a \mid \alpha \in \mathbb{R}\}.$$

Thus each V_a is a one-dimensional subspace of X and the map $a \mapsto x_1{}^a$ is continuous on A. By Theorem 2.5, for each $x \in X$, the parameter mapping $a \mapsto P_{V_a}(x)$ is upper semicontinuous on A.

Now let $x = (0, 1) \in X$. It is readily verified that for each $a \in A \backslash \{0\}$,

$$P_{V_a}(x) = \frac{\operatorname{sgn} a}{1 + | a |} x_1{}^a.$$

Thus $P_{V_a}(x) \to x_1{}^0$ as $a \searrow 0$ and $P_{V_a}(x) \to -x_1{}^0$ as $a \nearrow 0$. But

$$P_{V_0}(x) = \{\alpha x_1{}^0 \mid | \alpha | \leqslant 1\}.$$

It follows that the parameter map $a \mapsto P_{V_a}(x)$ is not lower semicontinuous at $a = 0$. Also, it is obvious that there can exist no selection for the parameter map which is continuous at $a = 0$.

This example suggests that to prove the existence of continuous selections for the parameter map $a \mapsto P_{V_a}(x)$, one must resort to alternate, more direct, methods rather than appealing to Micheal's selection theorem (which requires lower semicontinuity of the mapping in question). Indeed, in the next section we exhibit an example in which the parameter mapping is *not* lower semicontinuous and yet it *does* have a continuous selection.

3. Parametric Approximation by Weak Chebyshev Subspaces of $C[\alpha, \beta]$.

In this section we will be considering certain subspaces of the Banach space $C[\alpha, \beta]$, the real-valued continuous functions on the compact interval $[\alpha, \beta]$ in $\mathbb{R}$, and endowed with the supremum norm. We first show that the parameter mapping for a certain class of weak Chebyshev subspaces in $C[\alpha, \beta]$ admits a continuous selection.

3.1 Definitions. (1) An n-dimensional subspace V of $C[\alpha, \beta]$ is called *weak Chebyshev* if each $v \in V$ has at most $n - 1$ sign changes, i.e., there do not exist $n + 1$ points $\alpha \leqslant t_0 < t_1 < \cdots < t_n \leqslant \beta$ such that $v(t_i)\, v(t_{i+1}) < 0$ $(i = 0, 1,..., n)$.

(2) Let V be an n-dimensional subspace of $C[\alpha, \beta]$ and $x \in C[\alpha, \beta]$. An element $v \in P_V(x)$ is called an *alternation element* of x if there exist $n + 1$ points $\alpha \leqslant t_0 < t_1 < \cdots < t_n \leqslant \beta$ such that

$$\sigma(-1)^i(x - v)(t_i) = \| x - v \| \qquad (i = 0, 1,..., n)$$

for some $\sigma \in \{-1, 1\}$. The points $t_0, t_1,..., t_n$ are called *alternating extreme points* of $x - v$.

The following known result, connecting these concepts, will be useful to us.

3.2 Theorem. (1) (Jones and Karlovitz [4]) *An n-dimensional subspace V of $C[\alpha, \beta]$ is weak Chebyshev if and only if for each $x \in C[\alpha, \beta]$, there exists at least one alternating element in $P_V(x)$.*

(2) (Nürnberger and Sommer [10]) *Let V be an n-dimensional weak Chebyshev subspace of $C[\alpha, \beta]$. Then the following conditions are equivalent:*

(i) *Each nonzero $v \in V$ has at most n distinct zeros;*

(ii) *For each $x \in C[\alpha, \beta]$, there exists exactly one alternating element in $P_V(x)$.*

3.3 Theorem. *Let A be a topological space and for each $a \in A$ let V_a be an n-dimensional weak Chebyshev subspace of $C[\alpha, \beta]$ such that each nonzero $v \in V_a$ has at most n distinct zeros. Let $x \in C[\alpha, \beta]$ and*

$$\tilde{V}_a = \{v \in V_a \mid \| v \| \leqslant 2 \| x \|\}.$$

If the mapping $a \mapsto \tilde{V}_a$ is Hausdorff continuous on A, then there exists a continuous selection for the parameter mapping $a \mapsto P_{V_a}(x)$.

Proof. By Theorem 3.2 (2), for each $a \in A$, there exists exactly one alternating element $v_a \in P_{V_a}(x)$. We define a selection s for the parameter mapping by setting

$$s(a) = v_a \qquad (a \in A).$$

If s were not continuous at some $a_0 \in A$, there is a net (a_δ) in A with $a_\delta \to a_0$ such that $(s(a_\delta))$ is bounded away from $s(a_0)$. Since $\tilde{V}_a$ is compact, the same proof as given in Lemma 2.3 shows that the net $(s(a_\delta))$ has a subnet (which we may assume to be the net itself) converging to a point $v_0 \in P_{\tilde{V}_{a_0}}(x) = P_{V_{a_0}}(x)$. In particular, $v_0 \neq s(a_0)$. Let $\alpha \leqslant t_0^\delta < t_1^\delta < \cdots < t_n^\delta \leqslant \beta$ be $n + 1$ alternating extreme points for $x - s(a_\delta)$. Then

$$\sigma_\delta(-1)^i(x - s(a_\delta))(t_i^\delta) = \| x - s(a_\delta)\| \qquad (i = 0, 1,..., n)$$

for some $\sigma_\delta \in \{-1, 1\}$. By passing to a further subnet if necessary, we may assume that $t_i^\delta \to t_i$ $(i = 0, 1,..., n)$ and all the σ_δ are the same, say $\sigma_\delta = \sigma \in \{-1, 1\}$ for all δ. Taking limits we obtain

$$\sigma(-1)^i(x - v_0)(t_i) = \| x - v_0 \| \qquad (i = 0, 1,..., n).$$

Thus v_0 is an alternating element for x. By uniqueness of alternating elements, $v_0 = s(a_0)$, which is a contradiction. ∎

Now we give a simple application of Theorem 3.3 (which should be contrasted with Example 2.7).

3.4 Example. Let $A = [-1, 1]$ and for each $a \in A$, define $V_a = \operatorname{span}\{v_a\} \subset C[-1, 1]$, where $v_a(t) = | a - t |$ $(t \in [-1, 1])$. Then for any $x \in C[-1, 1]$, the hypothesis of Theorem 2.5 is fulfilled so the mapping $a \mapsto \tilde{V}_a$ is Hausdorff continuous on A. From Theorem 3.3, it follows that the parameter mapping $a \mapsto P_{V_a}(x)$ has a continuous selection.

Next we consider a particular parameter mapping which arises naturally in spline approximation.

3.5 Definition. Let $\alpha = a_0 < a_1 < \cdots < a_k < a_{k+1} = \beta$ be k fixed knots in $[\alpha, \beta]$. The class of *polynomial splines* of degree n with these k knots is defined by

$$\begin{aligned} S_{n,k}(a) &= S_{n,k}(a_1, a_2, ..., a_k) \\ &= \{v \in C[\alpha, \beta] \mid v \text{ has } n - 1 \text{ continuous} \\ &\qquad \text{derivatives in } [\alpha, \beta],\ v|_{[a_i, a_{i+1}]} \text{ is a} \\ &\qquad \text{polynomial of degree at most } n \quad (i = 0, 1,..., k)\}. \end{aligned}$$

Equivalently, $S_{n,k}(a_1, a_2, \ldots, a_k)$ is the $n + k + 1$-dimensional subspace of $C[\alpha, \beta]$ spanned by the basis vectors $\{1, t, t^2, \ldots, t^n, (t - a_1)_+^n, \ldots, (t - a_k)_+^n\}$, where

$$(t - a_i)_+^n = (t - a_i)^n \quad \text{if} \quad t \geqslant a_i$$
$$= 0 \quad \text{if} \quad t < a_i.$$

For the remainder of the paper A will denote the parameter set

$$A = \{a = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k \mid \alpha \equiv a_0 < a_1 < \cdots < a_k < a_{k+1} \equiv \beta\}$$

Fix any $x \in C[\alpha, \beta]$ and consider the following parameter mapping on A:

$$a \mapsto P_{S_{n,k}(a)}(x).$$

It is natural to ask how the set $P_{S_{n,k}(a)}(x)$ depends on the parameter a, i.e., on the knots. We will show that for *some* x the parameter mapping $a \mapsto P_{S_{n,k}(a)}(x)$ is not lower semicontinuous; however, in the case when $k \leqslant n + 1$, for any x there exists a continuous selection for this parameter mapping.

3.6 PROPOSITION. *For a fixed function $x \in C[\alpha, \beta]$, the mapping*

$$a \mapsto \tilde{S}_{n,k}(a) \equiv \{v \in S_{n,k}(a) \mid \| v \| \leqslant 2 \| x \|\}$$

is Hausdorff continuous on A. In particular, the parameter mapping

$$a \mapsto P_{S_{n,k}(a)}(x)$$

is upper semicontinuous on A.

Proof. By Theorem 2.5 it suffices to verify that the basis

$$\{1, t, \ldots, t^n, (t - a_1)_+^n, \ldots, (t - a_k)_+^n\}$$

of $S_{n,k}(a)$ varies continuously with $a \in A$. And for this, it clearly suffices to show that the mappings $a \mapsto (t - a_i)_+^n$ $(i = 1, 2, \ldots, k)$ are continuous on A. Fix any $a_1, b_1 \in [\alpha, \beta]$ with $a_1 < b_1$. Then if $\alpha \leqslant t \leqslant a_1$,

$$|(t - a_1)_+^n - (t - b_1)_+^n| = 0.$$

If $a_1 < t < b_1$, then

$$|(t - a_1)_+^n - (t - b_1)_+^n| = |(t - a_1)^n| \leqslant |b_1 - a_1|^n \leqslant C_1 |b_1 - a_1|$$

for some constant C_1 depending only on α, β, and n. If $b_1 \leqslant t \leqslant \beta$, then

$$|(t - a_1)_+^n - (t - b_1)_+^n| = |(t - a_1)^n - (t - b_1)^n| \leqslant C_2 | b_1 - a_1 |$$

for some constant C_2 depending only on α, β, and n. Thus

$$\|(t - a_1)_+^n - (t - b_1)_+^n\| \leqslant C | a_1 - b_1 |$$

for some constant C. Replacing a_1 by a_i, this shows that the mapping $a \mapsto (t - a_i)_+^n$ is continuous on A. ∎

In the sequel, we will refer to the following results which we state here for ease of reference.

3.7 THEOREM. (1) (Rice [12], Schumaker [14]) *A function* $v_0 \in S_{n,k}(a_1, a_2, ..., a_n)$ *is a best approximation to* $x \in C[\alpha, \beta]$ *if and only if* $x - v_0$ *has* $n + j + 1$ *alternating extreme points on some subinterval* $[a_i, a_{i+j}]$.

(2) (Rice [12]) *Let* $v_0 \in S_{n,k}(a_1, a_2, ..., a_k)$ *be a best approximation to* $x \in C[\alpha, \beta]$ *such that* $x - v_0$ *has* $n + j + 1$ *alternating extreme points in* $[a_i, a_{i+j}]$, *but does not have* $n + l + 1$ *alternating extreme points in any subinterval* $[a_r, a_{r+l}]$ *of* $[a_i, a_{i+j}]$. *Then all best approximations to* x *from* $S_{n,k}(a_1, a_2, ..., a_k)$ *coincide on* $[a_i, a_{i+j}]$.

(3) (Strauss [16]) *Let* $v_0 \in S_{n,k}(a_1, a_2, ..., a_k)$ *be a best approximation to* $x \in C[\alpha, \beta]$ *such that* $x - v_0$ *has at least* $j + 1$ *alternating extreme points in each interval*

$$\text{(i)} \quad [\alpha, a_j), (a_{k-j+1}, \beta] \qquad (j = 1, 2, ..., k)$$

and

$$\text{(ii)} \quad (a_i, a_{i+j+n}) \qquad (j \geqslant 1, k > n + 1).$$

(Note that condition (ii) is vacuously satisfied if $k \leqslant n + 1$.) Then v_0 is the *unique* best approximation of x from $S_{n,k}(a_1, a_2, ..., a_k)$.

3.8 PROPOSITION. *There exists a function* $x \in C[\alpha, \beta]$ *such that the parameter mapping*

$$a \mapsto P_{S_{n,k}(a)}(x)$$

is not lower semicontinuous.

Proof. Fix any $a^0 = (a_1, a_2, ..., a_k) \in A$. We will construct two functions $x \in C[\alpha, \beta]$ and $v_0 \in S_{n,k}(a^0)$ and a sequence (a^m) in A converging to a^0 having the following properties:

(1) $0, v_0 \in P_{S_{n,k}(a^0)}(x)$, $v_0 \neq 0$;

(2) For each m, $P_{S_{n,k}(a^m)}(x) = \{0\}$.

Having such functions, it follows that the parameter mapping $a \mapsto P_{S_{n,k}(a)}(x)$ is not lower semicontinuous at a^0 (because there exists no sequence (v_m), with $v_m \in P_{S_{n,k}(a^m)}(x)$ and $v_m \to v_0$).

Let

$$v_0(t) = \frac{-1}{(\beta - a_k)^n} (t - a_k)_+^n .$$

Then $v_0 \in S_{n,k}(a^0)$, $v_0(t) = 0$ for all $t \in [\alpha, a_k]$, and $v_0(\beta) = -1$. Letting $N = n + 2 + k$, it is easy to construct a function $x \in C[\alpha, \beta]$ which has the following properties:

(i) $\| x \| = 1$;

(ii) x is piecewise linear on the interval $[\alpha, a_{k-1}]$ and x has (at least) N alternating extreme points in each interval $(\alpha, a_1), (a_1, a_2), \ldots, (a_{k-2}, a_{k-1})$;

(iii) x has (at least) N alternating extreme points in $(a_{k-1}, \frac{1}{2}(a_{k-1} + a_k))$;

(iv)

$$\begin{aligned} x(t) &= 1 && \text{if} \quad t \in [\tfrac{1}{2}(a_{k-1} + a_k), a_k], \\ &= v_0(t) + 1 && \text{if} \quad t \in [a_k, \tfrac{1}{2}(a_k + \beta)], \\ &= -1 && \text{if} \quad t = \beta \end{aligned}$$

and x is linear for $t \in [\frac{1}{2}(a_k + \beta), \beta]$.

Then $\| x \| = 1 = \| x - v_0 \|$ and, since x and $x - v_0$ both have at least $N \geqslant n + 2$ alternating extreme points in $[\alpha, a_1]$, it follows by Theorem 3.7 (1) that 0 and v_0 are both in $P_{S_{n,k}(a^0)}(x)$.

For m sufficiently large (viz., $m > 1/(a_k - a_{k-1})$) we define $a_i^m = a_i$ $(i = 1, 2, \ldots, k-1)$, $a_k^m = a_k - 1/m$, and $a^m = (a_1^m, a_2^m, \ldots, a_k^m)$. Then $a^m \to a^0$ as $m \to \infty$. Since x has at least $N \geqslant \max\{n + 2, k + 1\}$ alternating extreme points in each interval $(\alpha, a_1^m), (a_1^m, a_2^m), \ldots, (a_{k-1}^m, a_k^m)$ and 2 alternating extreme points in $(a_k^m, \beta]$, it follows by (1) and (3) of Theorem 3.7 that $P_{S_{n,k}(a^m)}(x) = \{0\}$ for each m. This completes the proof. ∎

Despite this negative result, we can prove that the parameter map has a continuous selection when $k \leqslant n + 1$. In the proof of this result among others we use a selection method of Nürnberger and Sommer [10]. To shorten the proof, we refer several times to statements, which can be found in the proof of the Chracterization Theorem 3.1 in [10].

3.9 THEOREM. *If $k \leqslant n + 1$, then for each $x \in C[\alpha, \beta]$ there exists a continuous selection for the parameter mapping $a \to P_{S_{n,k}(a)}(x)$.*

Proof. Fix $x \in C[\alpha, \beta[$.

1. Construction of the selection:

Let $a = (a_1, ..., a_k) \in A$ and $v_0 \in P_{S_{n,k}(a)}(x)$. By Theorem 3.7 (1) and (2), there exists an interval $[a_l, a_{l+1}] \subset [\alpha, \beta]$, on which all $v \in P_{S_{n,k}(a)}(x)$ coincide.

(a) We first approximate $x - v_0$ in $[a_{l+1}, \beta]$ by $G_{l+1} = \text{span}\{(t - a_{l+1})_+^n, ..., (t - a_k)_+^n\}$. Since G_{l+1} is a $(k - l)$-dimensional weak Chebyshev subspace, by Theorem 3.2 (1), there exists an alternation element $v_1 \in P_{G_{l+1}}(x - v_0)$. Then $v_0 + v_1 \in P_{S_{n,k}(a)}(x)$.

(b) By [10] any two alternation elements $v_1, v_2 \in P_{G_{l+1}}(x - v_0)$ coincide on $[a_{l+1}, a_{l+2}]$.

(c) Furthermore from [10] it follows, that if $\bar{v}_0 \in P_{S_{n,k}(a)}(x)$ and $\bar{v}_1$ is an alternation element in $P_{G_{l+1}}(x - \bar{v}_0)$ by approximation in $[a_{l+1}, \beta]$ then $\bar{v}_0 + \bar{v}_1 = v_0 + v_1$ in $[a_{l+1}, a_{l+2}]$.

(d) We continue this method in the following way: We approximate $x - v_0 - v_1$ in $[a_{l+2}, \beta]$ by $G_{l+2} = \text{span}\{(t - a_{l+2})_+, ..., (t - a_k)_+\}$. By Theorem 3.2 (1), there exists an alternation element $v_2 \in P_{G_{l+2}}(x - v_0 - v_1)$. As in (1b) all these alternation elements coincide in $[a_{l+2}, a_{l+3}]$ and as in (1c) we have that $v_0 + v_1 + v_2 \in P_{S_{n,k}(a)}(x)$ is independent of the choice of v_0 and $v_0 + v_1$ in $[a_{l+2}, a_{l+3}]$. We continue this method up to the last interval $[a_k, \beta]$ and obtain a function $\bar{v} = v_0 + v_1 + \cdots + v_{k+l}$.

(e) Using the same kind of arguments as in (1c) and (d) we get a function $\bar{\bar{v}} = v_{-l} + v_{-l+1} + \cdots + v_0 \in P_{S_{n,k}(a)}(x)$, where for each i the function g_{-i} is an alternation element in $P_{\bar{G}_{l+1-i}}(x - v_{-i+1} - \cdots - v_0)$ by approximation with $\bar{G}_i = \text{span}\{(a_1 - t)_+, ..., (a_i - t)_+\}$ in $[\alpha, a_{l+1-i}]$, where

$$(a_i - t)_+^n = (a_i - t)^n \quad \text{if} \quad t \leqslant a_i,$$
$$0 \quad \text{if} \quad t > a_i.$$

We define $s(a) = v_{-l} + \cdots + v_{-1} + v_0 + v_1 + \cdots + v_{k-l}$ and have that $s(a) \in P_{S_{n,k}(a)}(x)$.

(f) By [10] $s(a)$ is independent of the choice of the interval $[a_l, a_{l+1}]$, on which all $v \in P_{S_{n,k}(a)}(x)$ coincide.

Therefore $a \to s(a)$ is a selection for the parameter mapping $a \to P_{S_{n,k}(a)}(x)$.

(2) We show that s is a continuous selection.

Assume that s is not continuous. Then there exists a point $a \in A$ and a sequence (a_m) in A such that $a_m \to a$ and $s(a_m)$ does not converge to $s(a)$, where $a_m = (a_1^m, ..., a_k^m)$.

(a) We show that $(s(a_m))$ has a subsequence converging to a function $v \in P_{S_{n,k}(a)}(x)$. For each m we can write $s(a_m) = \alpha_0^m + \alpha_1^m t + \cdots + \alpha_n^m t^n + \beta_1^m (t - a_1^m)_+^n + \cdots + \beta_k^m (t - a_k^m)_+^n$. By the proof of Proposition 3.6 $(t - a_i^m)_+^n \to (t - a_i)_+^n$, $i = 1, ..., k$. Thus (α_i^m), $i = 0, 1, ..., n$, and (β_i^m), $i = 1,\dots, k$, are bounded sequences, as been shown in the proof of Theorem 2.5. Therefore $(s(a_m))$ has a convergent subsequence, which for notational convenience we again denote by $(s(a_m))$, converging to a function $v \in S_{n,k}(a)$. We show that $v \in P_{S_{n,k}(a)}(x)$. If not, there exists a function $\bar{v} \in S_{n,k}(a)$ such that $\| x - v \| > \| x - \bar{v} \|$.

The function $\bar{v}$ can be written as $\bar{v} = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n + \beta_1(t - a_1)_+^n + \cdots + \beta_k(t - a_k)_+^n$. Then $\bar{v}_m = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n + \beta_1(t - a_1^m)_+^n + \cdots + \beta_k(t - a_k)_+^n$ is in $S_{n,k}(a_m)$ and converges to $\bar{v}$. Since $s(a_m) \to v$ we have for sufficiently large m that $\| x - s(a_m)\| > \| x - \bar{v}_m \|$. This is a contradiction to $s(a_m) \in P_{S_{n,k}(a_m)}(x)$.

This shows that we may assume that $a_m \to a$ and $s(a_m) \to v$, where $v \in P_{S_{n,k}(a)}(x)$, $v \neq s(a)$.

(b) We set $I(a) = \{t \in [\alpha, \beta]: v_1(t) = v_2(t)$ for each $v_1, v_2 \in P_{S_{n,k}(a)}(x)\}$. Passing again to a subsequence of (a_m), by Theorem 3.7 (a) and (2) we may assume that there exists an interval $[a_r, a_{r+j}]$ such that for each m $I(a_m) = [a_r^m, a_{r+j}^m]$ and $x - a_m$ has $n + j + 1$ alternating extreme points $a_r^m \leqslant z_0^m < z_1^m < \cdots < z_{n+j}^m \leqslant a_{r+j}^m$, i.e., $\epsilon(-1)^i(x - s(a_m))(z_i^m) = \| x - s(a_m)\|$, $i = 0, 1,\dots, n + j$, $\epsilon = \pm 1$. Since $[\alpha, \beta]$ is compact we may assume that $z_i^m \to z_i$, $i = 0, 1,\dots, n + j$. Then

$$\begin{aligned} \| x - v \| &= \lim_{m\to\infty} \| x - s(a_m)\| = \epsilon(-1)^i \lim_{m\to\infty} (x - s(a_m))(z_i^m) \\ &= \epsilon(-1)^i(x - v)(z_i), \end{aligned}$$

$i = 0, 1,\dots, n + j$. Therefore $x - v$ has $n + j + 1$ alternating extreme points in $[a_r, a_{r+j}]$. Thus by Theorem 3.7 (2) there exists an interval $[a_p, a_{p+i}] \subset [a_r, a_{r+j}]$ with $[a_p, a_{p+i}] \subset I(t)$.

(c) According to (2b) there exists an interval $[a_l, a_{l+1}] \subset I(a)$ such that $[a_l^m, a_{l+1}^m] \subset I(a_m)$ for each m. Since the selection s is independent of the choice of such intervals, the function $s(a)$ can be defined by starting with $[a_l, a_{l+1}]$ and the functions $s(a_m)$ can be defined by starting with $[a_l^m, a_{l+1}^m]$ for each m. Therefore $s(a) = v_{-l} + \cdots + v_{-1} + v_0 + v_1 + \cdots + v_{k-l}$ and $s(a_m) = v_{-l} + \cdots + v_{-1} + v_0^m + v_1^m + \cdots + v_{k-l}^m$. Since (v_i^m) is bounded for each i, by arguments similar to those in (2a) for each i the sequence (v_i^m) has a convergent subsequence, which we again denote by (v_i^m), converging to a function $\bar{v}_i$. Then of course we have $v = \bar{v}_{-l} + \cdots + \bar{v}_{-1} + \bar{v}_0 + \bar{v}_1 + \cdots + \bar{v}_{k-l}$. We show that for each i the functions v_i can be taken as $\bar{v}_i$, where the functions v_i are those which appear in the definition of $s(a)$. This is done by induction. We consider only the case $i \geqslant 0$, because the case $i < 0$ can be proved analogously. For $i = 0$ we have $v_0^m \in P_{S_{n,k}(a_m)}(x)$ and $v_0^m \to \bar{v}_0$ so that $\bar{v}_0 \in P_{S_{n,k}(a)}(x)$. Since all $v \in P_{S_{n,k}(a)}(x)$ coincide on $[a_l, a_{l+1}]$ and the selection is independent of the choice of any $v \in P_{S_{n,k}(a)}(x)$ we can take $v_0 = \bar{v}_0$. Now suppose we can take $v_j = \bar{v}_j$ for $0 \leqslant j < i$. For each

m the function v_i^m is an alternation element of $x - (v_0^m + \cdots + v_{i-1}^m)$ by approximation with G_{l+i}^m in $[a_{l+i}^m, \beta]$. Using arguments similar to those in (2a) we get that, since $v_j^m \to \bar{v}_j$, $j = 0, 1,..., i$, the function $\bar{v}_i$ is an alternation element of $x - (v_0 + \cdots + v_{i-1})$ by approximation with G_{l+i} in $[a_{l+i}, \beta]$. We note that, since $v_j^m \to \bar{v}_j$, $j = 0, 1,..., i$, the alternation properties of $x - (v_0^m + \cdots + v_{i-1}^m) - v_i^m$ carry over to $x - (v_0 + \cdots + v_{i-1}) - \bar{v}_i$. From (1b)–(e) it follows that $\bar{v}_0 + \cdots + \bar{v}_i = v_0 + \cdots + v_i$ in $[a_{l+i}, a_{l+i+1}]$ and thus we can take $v_i = \bar{v}_i$. Therefore $s(a) = v$,a contradiction. This completes the proof. ∎

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